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On the convergence of solutions of nonautonomous functional differential equations

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Abstract

In this paper we study the asymptotic behaviour of solutions of delay differential equations when the right hand side of equation can be estimated by the maximum function using a new method based on the Liapunov-Razumikhin principle, differential inequalities and an invariance principle. This method can be applicable for nonautonomous equations without local Lipschitz property.

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1. Introduction

We investigate the asymptotic behaviour of solutions of delay differential equations when the right hand side of equation can be estimated by the maximum function. The results in this study only concern autonomous or periodic equations or they use linear maximum estimate. For using linear maximum estimate it is necessary that the right hand side of equation has the local Lipschitz property [1,2,3,7].

In the present paper we develop a new method applicable for nonautonomous equations without local Lipschitz property. The method is based on the Liapunov-Razumikhin principle, differential inequalities and an invariance principle.

2. Preliminaries

Let $r > 0$. We define

$$C_r = \{\phi : [-r, 0] \to \mathbb{R}, \phi \text{ is continuous}\},$$

$$M(\phi) = \max\{\phi(s) : s \in [-r, 0]\},$$

$$m(\phi) = \min\{\phi(s) : s \in [-r, 0]\},$$

$$\|\phi\| = M(|\phi|).$$

For $\alpha \in \mathbb{R}$, we introduce

$$T(\alpha) = \{\phi \in C_r : \phi(0) = \alpha,\ \phi(s) < \alpha,\ s \in [-r, 0)\},$$

$$t(\alpha) = \{\phi \in C_r : \phi(0) = \alpha,\ \phi(s) > \alpha,\ s \in [-r, 0)\}.$$

For $\alpha \in \mathbb{R}$ and ϵ_1, $\epsilon_2 > 0$, we define

$$S(\alpha,\epsilon_1,\epsilon_2) = \{\phi \in C_r : \alpha-\epsilon_1 \le \phi(0) \le \alpha, \alpha \le M(\phi) < \alpha+\epsilon_2\},$$

$$s(\alpha,\epsilon_1,\epsilon_2) = \{\phi \in C_r : \alpha \le \phi(0) \le \alpha+\epsilon_1,\ \alpha-\epsilon_2 < m(\phi) \le \alpha\}.$$

For $\alpha \in \mathbb{R}$, let $H(\alpha)$ be the set of continuous functions $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ such that $h(\alpha,\alpha) = 0$, and the solutions of the initial value problem $u'(t) = h(u(t),\alpha)$, $u(0) = \alpha$ satisfy the left side uniqueness condition. That is, if $u : (t_1,t_2) \to \mathbb{R}$, $t_1 < 0 < t_2$, $u(0) = \alpha$, u is differentiable and satisfies equation $u'(t) = h(u(t),\alpha)$ for $t \in (t_1,t_2)$, then $u(t) \equiv \alpha$ for all $t \in (t_1, 0]$.
Consider the equation

$$(1) \qquad x'(t) = f(t,x_t),$$

where $f : [0,\infty) \times C_r \to \mathbb{R}$ is continuous.
For $A > 0$, $x : [-r,A) \to \mathbb{R}$ is a solution of Eq. (1), if x is continuous, it is differentiable on $(0,A)$ and satisfies Eq. (1) on $(0,A)$.
It is known that if $\phi \in C_r$, then there are $A > 0$ and $x : [-r,A) \to \mathbb{R}$ such that x is a solution of Eq. (1) and $x(s) = \phi(s)$, $s \in [-r,0]$.
We denote by $X(\phi)$ the set of solutions x of Eq.(1) existing on $[-r,\infty)$ with $x_0 = \phi$.
Let $X = \cup_{\phi\in C_r} X(\phi)$.

Claim 1. *If* $f(t,\phi) \le 0$ *for all* $t \ge 0$, $\alpha \in \mathbb{R}$, $\phi \in T(\alpha)$, *and* $x \in X(\phi)$, *then* $M(x_t)$ *is monotone nonincreasing. [4]*

Claim 2. *If* $f(t,\phi) \ge 0$ *for all* $t \ge 0$, $\alpha \in \mathbb{R}$, $\phi \in t(\alpha)$, *and* $x \in X(\phi)$, *then* $m(x_t)$ *is monotone nondecreasing. [4]*

Theorem 1. *Assume that for every* $\alpha \in \mathbb{R}$ *there are* ϵ_1, $\epsilon_2 > 0$ *and* $h \in H(\alpha)$ *such that*
if $\phi \in S(\alpha,\epsilon_1,\epsilon_2)$ *and* $t \ge 0$, *then* $f(t,\phi) \le h(\phi(0), M(\phi))$.
Then for all $x \in X$ *either* $\lim_{t\to\infty} x(t)$ *exists in* $\mathbb{R}$ *or* $\lim_{t\to\infty} x(t) = -\infty$.

Theorem 2. *If the assumption of Theorem 1 is true and for every* $\phi \in t(\alpha)$, $t \ge 0$, $f(t,\phi) \ge 0$, *then for every* $x \in X$, $\lim_{t\to\infty} x(t)$ *exists in* $\mathbb{R}$.

Theorem 3. *Assume that for every* $\alpha \in \mathbb{R}$ *there are* ϵ_1, $\epsilon_2 > 0$ *and* $h \in H(\alpha)$ *such that*
if $\phi \in s(\alpha,\epsilon_1,\epsilon_2)$ *and* $t \ge 0$, *then* $f(t,\phi) \ge h(\phi(0), m(\phi))$.
Then for all $x \in X$ *either* $\lim_{t\to\infty} x(t)$ *exists in* $\mathbb{R}$ *or* $\lim_{t\to\infty} x(t) = \infty$.

Theorem 4. *If the assumption of Theorem 3 is true and for every* $\phi \in T(\alpha)$, $t \ge 0$, $f(t,\phi) \le 0$, *then for every* $x \in X$, $\lim_{t\to\infty} x(t)$ *exists in* $\mathbb{R}$.

Theorem 5. *Consider the following assumptions:*

i) *For every* $\alpha \in \mathbb{R}$, $t \ge 0$,
$\phi \in T(\alpha)$ *implies* $f(t,\phi) \le 0$,
$\phi \in t(\alpha)$ *implies* $f(t,\phi) \ge 0$.

ii) *For every* $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ *there are* ϵ_1, $\epsilon_2 > 0$ *and* $h \in H(\alpha)$, $g \in H(\beta)$ *such that either*
$\phi \in s(\alpha, \epsilon_1, \epsilon_2)$, $t \geq 0$ *implies* $f(t, \phi) \geq h(\phi(0), m(\phi))$ *or*
$\phi \in S(\beta, \epsilon_1, \epsilon_2)$, $t \geq 0$ *implies* $f(t, \phi) \leq g(\phi(0), M(\phi))$.

Then for every $x \in X$, $\lim_{t\to\infty} x(t)$ exists in $\mathbb{R}$.

3. The proof of Theorem 1.

If $\lim_{t\to\infty} M(x_t) = -\infty$, then $\lim_{t\to\infty} x(t) = -\infty$. Suppose that $\lim_{t\to\infty} M(x_t) = \alpha \in \mathbb{R}$. Since $T(\alpha) \subset S(\alpha, \epsilon_1, \epsilon_2)$ for every $\alpha \in \mathbb{R}$, ϵ_1, $\epsilon_2 > 0$, and $h(\alpha, \alpha) = 0$, Claim 1 implies that $M(x_t)$ is monotone nonincreasing, so $M(x_t) \geq \alpha$ for all $t \in [0, \infty)$. By way of contradiction, assume that $\lim_{t\to\infty} x(t)$ does not exist. Then there is $\beta < \alpha$ such that $\liminf x(t) = \beta$. For this α let $h \in H(\alpha)$ and choose ϵ_1, $\epsilon_2 > 0$ such that $\beta < \alpha - \epsilon_1$ and $f(t, \phi) \leq h(\phi(0), M(\phi))$ for all $t \geq 0$, $\phi \in S(\alpha, \epsilon_1, \epsilon_2)$. Then there is $T > 0$ such that $M(x_t) < \alpha + \epsilon_2$ for all $t \geq T$. There is a sequence (t_n) such that $t_n \to \infty$ and $x(t_n) = \alpha - \epsilon_1$. As $M(x_{t_n}) \geq \alpha$, there are t'_n, t''_n such that $t_n \leq t'_n < t''_n \leq t_n + r$, $x(t'_n) = \alpha - \epsilon_1$, $\alpha - \epsilon_1 < x(t) < \alpha$, $t \in (t'_n, t''_n)$, and $x(t''_n) = \alpha$. Then, for all $t \in (t'_n, t''_n)$, we have $x'(t) = f(t, x_t) \leq h(x(t), M(x_t)) \leq \max\{h(x, y) : \alpha - \epsilon_1 \leq x \leq \alpha + \epsilon_2,\ \alpha \leq y \leq \alpha + \epsilon_2\}$, that is $x'(t)$ is bounded on (t'_n, t''_n). So there is $\epsilon > 0$ such that $\epsilon \leq t''_n - t'_n \leq r$. We can suppose that there is r^* such that $t''_n - t'_n \to r^*$ and $\epsilon \leq r^* \leq r$. Consider for all $n \in \mathbb{N}$ the initial value problem: $u'_n(s) = h(u_n(s), M(x_{t_n+s}))$, $s \geq 0$, $u_n(0) = \alpha - \frac{\epsilon_1}{2}$. For all $n \in \mathbb{N}$, there are $A_n > 0$ and $u_n : [0, A_n) \to \mathbb{R}$ such that u_n is a solution of this initial value problem. We have $x(t'_n) < u_n(0)$ and $x'(t'_n + s) \leq h(x(t'_n + s), M(x_{t'_n+s}))$, $s \geq 0$.

Using Theorem 1.2.1 and Remark 1.2.1 of [5] or Lemma 1.2 of [2], it follows $x(t'_n + s) < u_n(s)$, $s \in [0, A_n)$.

Therefore, for all $n \in \mathbb{N}$, there is b_n such that $\epsilon < b_n \leq r^*$, $u_n(b_n) = \alpha$, and $u_n(s) < \alpha$ for all $s \in [0, b_n)$. Since $(u_n(s))$ is equicontinuous and uniformly bounded on $[0, r^*]$, we can suppose that $(u_n(s))$ converges uniformly to $u(s)$ as $n \to \infty$, and $b_n \to b \in [0, r^*]$. Following the method of limit equation of nonautonomous differential equation presented in Appendix A of [6], we have $u_n(s) = \alpha - \frac{\epsilon_1}{2} + \int_0^s h(u_n(z), M(x_{t_n+z}))dz$, for every $0 \leq s \leq r^*$. Hence, letting $n \to \infty$, we get, $u(s) = \alpha - \frac{\epsilon_1}{2} + \int_0^s h(u(z), \alpha)dz$, for every $0 \leq s \leq r^*$. Obviously that $u(s)$ satisfies the properties $u(t) < \alpha$, $0 \leq t < b$, and $u(b) = \alpha$. Then the function $v(t) = u(b + t)$, $t \leq 0$ contradicts $h \in H(\alpha)$. The proof of Theorem 1 is complete. The proofs of Theorems 2-5 can be made analogously.

4. Application

Consider the equation

$$x'(t) = -l(x(t)) + a(t)l(x(t - r_1(t)) + b(t)l(x(t - r_2(t))),$$

where $l : \mathbb{R} \to \mathbb{R}$ is a continuous and nondecreasing function, a, b, r_1, $r_2 : [0, \infty) \to \mathbb{R}$ are continuous such that $a(t)$, $b(t) \geq 0$, $a(t) + b(t) = 1$, $0 \leq r_1(t)$, $r_2(t) \leq r$ for some $r \in (0, \infty)$ and for all $t \in [0, \infty)$. Suppose that for every α, $\gamma \in \mathbb{R}$ such that $\gamma < \alpha$, the improper integral $\int_{\gamma}^{\alpha} \frac{dx}{l(\alpha)-l(x)}$ does not exist. This condition implies that left hand side uniqueness is true for the initial value problem $u'(t) = -l(u(t)) + l(\alpha)$, $u(0) = \alpha$, that is $h(x,y) := -l(x) + l(y) \in H(\alpha)$, for every $\alpha \in \mathbb{R}$. The proof can be done in the same way as that of Osgood's uniqueness theorem [8]. If $\phi \in T(\alpha)$, then $\phi(0) = \alpha$ and $\phi(s) < \alpha$, for every $s \in [-r, 0]$, therefore $f(t,\phi) = -l(\phi(0)) + a(t)l(\phi(-r_1(t))) + b(t)l(\phi(-r_2(t))) \leq -l(\alpha) + a(t)l(\alpha) + b(t)l(\alpha) = 0$. Similarly, if $\phi \in t(\alpha)$, then $\phi(0) = \alpha$ and $\phi(s) > \alpha$, for every $s \in [-r, 0]$, therefore $f(t,\phi) \geq -l(\alpha) + a(t)l(\alpha) + b(t)l(\alpha) = 0$. Choosing α, $\beta \in \mathbb{R}$, $\alpha < \beta$, ϵ_1, $\epsilon_2 > 0$ and $\phi \in s(\alpha, \epsilon_1, \epsilon_2)$, then $\alpha = \phi(0) \geq m(\phi)$, so $f(t,\phi) \geq -l(\phi(0)) + a(t)l(m(\phi)) + b(t)l(m(\phi)) = -l(\phi(0)) + l(m(\phi)) = h(\phi(0), m(\phi)) \in H(\alpha)$. Choosing $\alpha, \beta \in \mathbb{R}, \alpha < \beta, \epsilon_1, \epsilon_2 > 0$ and $\phi \in S(\beta, \epsilon_1, \epsilon_2)$, then $M(\phi) \geq \phi(0) = \beta$, therefore $f(t,\phi) \leq -l(\phi(0)) + l(M(\phi)) = h(\phi(0), M(\phi)) \in H(\beta)$. Then, by Theorem 5 it follows that $\lim_{t\to\infty} x(t)$ exists in $\mathbb{R}$, where $x(t)$ is an arbitrary solution of the equation.

Particularly, if $l(x)$ has the form $l(x) = -(x - \frac{k}{e}) \log(\frac{k}{e} - x) + \frac{k}{e}$, if $\frac{k-1}{e} < x < \frac{k}{e}$, $k \in \mathbb{Z}$ and $l(\frac{k}{e}) = \frac{k}{e}$, then $l(x)$ does not satisfy the local Lipschitz property at $x = \frac{k}{e}$, but $\int_{\gamma}^{\alpha} \frac{dx}{l(\alpha)-l(x)}$ does not exist, if α, $\gamma \in \mathbb{R}$ such that $\gamma < \alpha$. Since $l'(x)$ is continuous on $\mathbb{R} \setminus \{\frac{k}{e} : k \in \mathbb{Z}\}$, it is sufficient to calculate the integral $I := \int_{\gamma}^{\frac{k}{e}} \frac{ds}{l(\frac{k}{e})-l(s)}$, where $\frac{k-1}{e} < \gamma < \frac{k}{e}$. As $\int_{\gamma}^{\frac{k}{e}} \frac{ds}{l(\frac{k}{e})-l(s)} = \lim_{\gamma_1 \to \frac{k}{e}-} \int_{\gamma}^{\gamma_1} \frac{ds}{(s-\frac{k}{e})\log(\frac{k}{e}-s)} =$

$\lim_{\gamma_1 \to \frac{k}{e}-} \log(-\log(\frac{k}{e} - \gamma_1)) - \log(-\log(\frac{k}{e} - \gamma)) = \infty$, I does not exist.

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